A Superfluid at All Temperatures

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We construct unitary, local, 3+1-dimensional conformal field theories with finitely many fields whose thermal state breaks a continuous global symmetry at every temperature. They are finite-N versions of the Chaudhuri–Choi–Rabinovici theories, whose planar circle of fixed points has an ordered arc. The circle is lifted at order $1/N^2$, and an infinite family of fixed points on this arc reaches arbitrarily weak coupling, where the superfluid is under parametric control. We also show that weakly gauging the symmetry gives a superconductor over an exponential range of temperatures.

Introduction.—Ordered phases of matter usually melt when they are heated. The equilibrium state minimizes the free energy $E - TS$, and at high temperature the entropy of disordered configurations dominates. In relativistic field theory this expectation is known as symmetry restoration [1–3]. Weinberg pointed out that it is not a theorem [3]; with several scalar multiplets, a negative cross-coupling can make one of the thermal masses negative while the potential remains bounded from below, such that a symmetry stays spontaneously broken at arbitrarily high temperature. This possibility has been explored extensively, for instance in cosmology [4–7], but two major obstacles have accompanied it. The scalar theories originally considered are not ultraviolet complete: their couplings grow at short distances, so they make sense only up to a cutoff scale, and hence a maximal temperature [8]. Furthermore, the effect results from a competition between loop contributions of opposite sign, which higher-order corrections might undo [9–11].

Conformal field theories (CFTs) remove the first obstacle. On $\mathbb{R}^3 \times S^1_\beta$ the temperature $T = 1/\beta$ is the only scale. If the thermal state of a CFT breaks a symmetry at one temperature, it breaks it at all temperatures. This observation motivated the study of thermal order in CFTs [12, 13], which is now understood as an instance of entropic order [14]: ordering one set of degrees of freedom allows another set to fluctuate more, so that the ordered state has the larger entropy. As emphasized in [15], in quantum field theory symmetry breaking at all temperatures had previously been established only in a non-integer number of dimensions, with non-local interactions, or in the strict planar limit. Each of these idealizations gives up a standard property of quantum field theory. Scalar theories in $d = 4 - \epsilon$ are not unitary [16], long-range models [17, 18] are not local, and the planar limit has infinitely many fields at each spacetime point. The last idealization is not innocent: conclusions drawn at $N = \infty$ can violate constraints that hold at any finite N, because the large-N limit need not commute with the long-distance limit [19, 20]. In the two-dimensional $SU(N)$ Thirring model the $1/N$ expansion appears to predict spontaneous breaking of the continuous chiral symmetry, which is impossible [21]; instead, the correlator of the order parameter decays as $|x|^{-1/N}$, which becomes long-range order only at $N = \infty$ [19]. All idealizations were first avoided in [15], where a unitary, local CFT with finitely many fields breaks its discrete $\mathbb{Z}_2$ symmetry spontaneously at all temperatures in $2 + 1$ dimensions, with control at large but finite N; see also [22–26].

Breaking a continuous symmetry at nonzero temperature requires going to $3 + 1$ dimensions, where the thermal state lives on $\mathbb{R}^3$. Chaudhuri, Choi, and Rabinovici (CCR) constructed weakly coupled conformal gauge theories whose planar limit has a circle of fixed points, an arc of which breaks a $U(1)$ symmetry at all temperatures [27, 28]. Whether an ordered fixed point survives at finite N was left open [29].

In this Letter we show that ordered fixed points do survive at finite N. The circle is lifted at order $1/N^2$ and leaves behind a discrete set of fixed points, whose positions on the circle depend on the ranks of the gauge groups and the numbers of flavors. These can be chosen so that a fixed point lies on the ordered arc. We construct an infinite family of such theories, in which the ranks grow and all couplings become arbitrarily weak. In this family the existence of an ordered fixed point, and the fact that its thermal state is a relativistic superfluid at every temperature, are under parametric perturbative control. We also show that coupling the broken symmetry to a weakly coupled photon gives a superconductor over an exponentially large range of temperatures [32].

The theories.—The gauge group is $G_1 \times G_2$, with $G_i = SU(N_i)_\mathrm{a} \times SU(N_i)_\mathrm{b}$ and $N_2 < N_1$. The sectors $i = 1, 2$ each contain $N_{f,i}$ Dirac fermions ψ_i in the fundamental of $SU(N_i)_\mathrm{a}$, $N_{f,i}$ Dirac fermions χ_i in the fundamental of $SU(N_i)_\mathrm{b}$, and a complex scalar Φ_i in the bifundamental. A $\mathbb{Z}_2$ symmetry exchanges the two factors and equates their gauge couplings to a common value g_i, and all Yukawa couplings are forbidden by flavor symmetries. The two sectors interact only through the scalar potential

$$V = \sum_{i=1,2} \left[\tilde{h}_i \operatorname{tr}\left(\Phi_i^\dagger \Phi_i\right)^2 + \tilde{f}_i \left(\operatorname{tr} \Phi_i^\dagger \Phi_i\right)^2 \right] + 2\tilde{\zeta} \operatorname{tr} \Phi_1^\dagger \Phi_1 \operatorname{tr} \Phi_2^\dagger \Phi_2 \,. \tag{1}$$

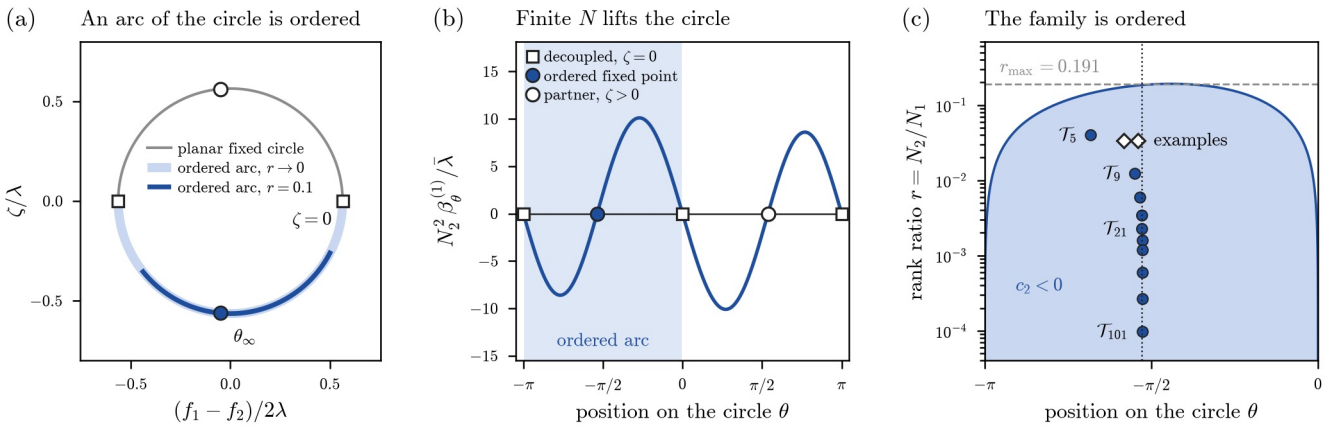


FIG. 1. From the planar conformal manifold to ordered CFTs at finite N. (a) The circle of planar fixed points (4) in the plane of the double-trace couplings $(f_1 - f_2)/2$ and ζ, in units of λ. Its lower half, where $\zeta < 0$, is ordered ($c_2 < 0$) for $r = N_2/N_1 \to 0$. At $r = 0.1$ the ordered arc is shorter, and it shrinks to a point at $r = r_{\max} = 0.191$. The markers show the points that survive at finite N in the family $\mathcal{T}_n$ as $n \to \infty$: the decoupled points with $\zeta = 0$ (squares), the ordered fixed point at θ_∞ (filled circle), and its partner at $-\theta_\infty$, which has $\zeta > 0$ and is not ordered (open circle). (b) The β-function $\beta_\theta^{(1)}$ (6) for the motion along the old circle at finite N, for $\mathcal{T}_{21}$. Its zeros are the fixed points that survive. The shaded band is the ordered arc for $r = 1/441$. (c) The ordered region $c_2 < 0$ in the plane of θ and r (shaded), with the two-loop fixed points of the family $\mathcal{T}_n$ for $5 \le n \le 101$, at $r = 1/n^2$ (circles), and of the theories $(N_1, N_{f,1}, N_2, N_{f,2}) = (150, 783, 5, 26)$ and $(300, 1562, 10, 52)$, at $r = 1/30$ (diamonds, right and left). The dotted line marks θ_∞.

We use the 't Hooft couplings $\lambda_i = g_i^2 N_i/16\pi^2$, $h_i = N_i \tilde{h}_i/16\pi^2$, $f_i = N_i^2 \tilde{f}_i/16\pi^2$, and $\zeta = N_1 N_2 \tilde{\zeta}/16\pi^2$.

The symmetry of interest is $U(1)_{\Phi_2}\colon \Phi_2 \to e^{i\alpha}\Phi_2$. It is anomaly-free and acts faithfully on gauge-invariant operators, since $\bar{\psi}_2 \Phi_2 \chi_2$ has charge one. Its gauge-invariant order parameter $\det \Phi_2$ has charge N_2.

Each sector has a Banks–Zaks fixed point [34, 35] when $x_i \equiv N_{f,i}/N_i$ is slightly below $21/4$. With $m_i \equiv 21N_i - 4N_{f,i}$, the two-loop value is

$$\lambda_i^* = \frac{21 - 4x_i}{26x_i - 54 - 3(4 + 2x_i)/N_i^2} \simeq \frac{m_i}{82.5\, N_i}\,. \qquad (2)$$

The quartic couplings first enter the β-functions of the gauge couplings at three loops, such that we can treat λ_1^* and λ_2^* as fixed in the β-functions of the quartic couplings.

The planar circle.—In the planar limit the single-trace couplings sit at $h_i = (3 - \sqrt{6})\lambda_i/4$. When $\lambda_1 = \lambda_2 = \lambda$, the double-trace couplings can be assembled into a symmetric matrix F with diagonal entries f_1, f_2 and off-diagonal entry ζ, obeying [27]

$$\beta_F = \big(12h^2 + \tfrac{9}{2}\lambda^2\big)\mathbf{1} + (16h - 12\lambda)F + 4F^2\,. \qquad (3)$$

This is covariant under $F \to OFO^T$ with $O \in O(2)$, which is an accidental symmetry of planar double-trace dynamics that rotates $\operatorname{tr}\Phi_1^\dagger\Phi_1$ into $\operatorname{tr}\Phi_2^\dagger\Phi_2$. The fixed points form an S^1 conformal manifold

$$\tfrac{1}{2}(f_1 + f_2) = \tfrac{\sqrt{6}}{2}\,\lambda\,, \qquad \tfrac{1}{2}(f_1 - f_2) + i\zeta = \widehat{R}\lambda\, e^{i\theta}\,, \quad (4)$$

with $\widehat{R} = [(18\sqrt{6} - 39)/16]^{1/2} \approx 0.564$ [Fig. 1(a)]. The circle persists to all orders in planar perturbation theory [27], as follows from the fact that planar double-trace β-functions are quadratic in the double-trace couplings, with coefficients fixed by the single-trace data [36, 37].

Whether the thermal state is ordered is determined by the effective potential of the static scalar modes on $\mathbb{R}^3 \times S^1_\beta$. At leading order it is $V + 16\pi^2 T^2 \sum_i c_i \operatorname{tr}\Phi_i^\dagger\Phi_i$, with one-loop thermal masses c_i linear in the couplings. Since T is the only scale, a negative c_i, with V being bounded from below, destabilizes the symmetric point at every temperature. For $h_i > 0$, V is bounded if and only if $a_i \equiv h_i + f_i > 0$ and $\zeta > -\sqrt{a_1 a_2}$, which holds on the entire S^1. On the circle

$$\frac{c_2}{\lambda} = \frac{1}{2} + \frac{\widehat{R}}{6}\Big(\frac{\sin\theta}{r} - \cos\theta\Big)\,, \qquad r \equiv \frac{N_2}{N_1}\,, \qquad (5)$$

while $c_1/\lambda = \frac{1}{2} + \frac{\widehat{R}}{6}(\cos\theta + r\sin\theta) > 0$. The β-functions do not depend on r, but c_2 does: each of the N_1^2 complex components of Φ_1 contributes $T^2/12$ to $\langle \operatorname{tr}\Phi_1^\dagger\Phi_1\rangle$, and the double-trace coupling turns this into a mass for Φ_2 enhanced by N_1/N_2. Thus $c_2 < 0$ on the arc $-\sin\theta > r(3/\widehat{R} - \cos\theta)$, where $\zeta < 0$, which exists for $r < 0.19144$ [27]. On this ordered arc Φ_2 condenses at every temperature.

Lifting at order $1/N^2$.—At finite N the accidental $O(2)$ is broken in two ways: by non-planar terms in the β-functions, and by the mismatch $\delta = \lambda_1^* - \lambda_2^*$ between the Banks–Zaks couplings, which are rational numbers determined by $(N_i, N_{f,i})$. There is no correction of or-

der $1/N$. Cutting a diagram at its double-trace vertices decomposes it into single-trace blocks. A block with k insertions of $\mathrm{tr}\,\Phi_i^\dagger\Phi_i/N_i$ scales as N_i^{2-k}, with a further N_i^{-2} for each handle, and since every double-trace vertex supplies two insertions, blocks with odd k occur in pairs. The ranks therefore enter only through $1/N_i^2$ and $1/(N_1N_2)$, as the explicit one- and two-loop β-functions confirm (End Matter). When δ is of order $\bar\lambda/N_2^2$, with $\bar\lambda \equiv (\lambda_1^*+\lambda_2^*)/2$, the conformal manifold is therefore lifted at order $1/N^2$, the first order beyond the planar limit.

Near the circle, the transverse components of the finite-N β-functions can be set to zero by shifting the couplings slightly off the circle. No analogous shift can eliminate the component tangent to the circle. The surviving fixed points are therefore determined by the zeros of $\beta_\theta^{(1)}$, the leading large-N contribution to the angular β-function $\beta_\theta = \frac{d\theta}{d\log\mu}$, which governs the RG flow along the circle (End Matter). We find

$$\beta_\theta^{(1)} = \frac{\bar\lambda}{2\widehat{R}N_2^2}\,\sin\theta\,\big[\Delta + g_r(\theta)\big]\,, \qquad \Delta \equiv \kappa\,\frac{\delta\,N_2^2}{\bar\lambda}\,, \quad (6)$$

with $\kappa = (18\sqrt{6}-39)/2 \approx 2.545$ and g_r given in Eq. (19). For $r \to 0$, $g_0 = \kappa\cos^2\theta + b\cos\theta + a$, with $a \approx 29.97$ and $b \approx -20.94$; it increases monotonically from 11.58 at $\theta = 0$ to 53.45 at $\theta = \pm\pi$. The zeros at $\sin\theta = 0$ are the decoupled theories with $\zeta = 0$. Interacting fixed points exist if and only if

$$-53.45 < \Delta < -11.58\,. \quad (7)$$

In that case, there are two solutions at $\theta = \pm\theta_*$ [Fig. 1(b)], where the direction along the old circle is irrelevant, with scaling dimension $4 + O(\bar\lambda/N_2^2)$. The one with $\zeta < 0$ lies on the ordered arc.

The integers as a tuning knob.—CCR achieved $\lambda_1^* = \lambda_2^*$ in the planar limit by setting $x_1 = x_2$. At finite N no such tuning is available, but none is needed. Eq. (2) gives

$$\Delta \simeq \kappa\left[\frac{N_2^3}{m_2}\Big(\frac{m_1}{N_1} - \frac{m_2}{N_2}\Big) - \frac{87}{165}\right], \quad (8)$$

which can be placed in the window (7) by a choice of integers. Since m_1 changes in steps of four, this requires $N_1 \gtrsim 0.24\,N_2^3/m_2$. The granularity of the integers, rather than the bound $r < 0.19$, forces the large sector to be much larger than the small one. Two explicit examples are $(N_1, N_{f,1}, N_2, N_{f,2}) = (150, 783, 5, 26)$ and $(300, 1562, 10, 52)$, with λ_i^* between 1.4×10^{-3} and 2.6×10^{-3}. The fixed points of their one- and two-loop β-functions, exact in N_i, are ordered (Table I).

An infinite family.—The bound $N_1 \gtrsim 0.24\,N_2^3/m_2$ suggests $N_2 = n$ and $N_1 = n^3$. Integrality of $N_{f,1}$ and $N_{f,2}$ then leads to the theories $\mathcal{T}_n$,

$$\begin{aligned} N_1 &= n^3\,, \quad & N_{f,1} &= \tfrac{1}{4}\big(21n^3 - n^2 + 12\big)\,, \\ N_2 &= n\,, \quad & N_{f,2} &= \tfrac{1}{4}\big(21n - 1\big)\,, \end{aligned} \quad (9)$$

with $n \equiv 1 \pmod 4$. Along this family, the coupling $\lambda_i^* \simeq 2/(165n)$ becomes parametrically small and $\Delta \to \Delta_\infty = -\frac{689}{55}\kappa \approx -31.89$, which sits inside the allowed window. The interacting fixed point approaches $\theta_\infty \approx -1.662$, where $h_i/\bar\lambda \to 0.138$, $f_1/\bar\lambda \to 1.174$, $f_2/\bar\lambda \to 1.276$, and $\zeta/\bar\lambda \to -0.562$ [Fig. 1(c)]. For ten members with $5 \le n \le 101$ we have verified that the one- and two-loop β-functions, exact in N_i, have this fixed point, with a potential bounded below, $c_1 > 0$, and $c_2 < 0$ (six of them are listed in Table I).

The family turns these results into statements that hold for all sufficiently large n. Along it all couplings are of order $1/n$, $r = 1/n^2$, and the terms that break the $O(2)$ are of order $1/n^4$ at one loop; since the circle has radius $\widehat{R}\bar\lambda$, they give $\beta_\theta^{(1)}$ of order $1/n^3$. Every further loop adds at least one power of $1/n$, and since the ranks enter only through $1/N_i^2$ and $1/(N_1N_2)$, no inverse powers of r appear. Hence

$$\beta_\theta = \frac{1}{n^3}\Big[\frac{1}{165\widehat{R}}\sin\theta\,\big(\Delta_\infty + g_0(\theta)\big) + O(1/n)\Big]. \quad (10)$$

Since g_0 is monotonic, the leading term has a simple zero at θ_∞, and a simple zero cannot be removed by a small perturbation. The transverse β-functions, whose Jacobian has eigenvalues of order $1/n$, can be solved near the circle, and for all sufficiently large n the full β_θ then has a zero near θ_∞. This is the same argument that establishes the Banks–Zaks fixed point at large N.

It is essential that the argument applies to the entire family. For each individual theory the existence of the ordered fixed point rests on numerical evidence, as it does *e.g.* for the Banks–Zaks fixed point of $SU(3)$ with sixteen flavors. Here, it is more delicate than usual: the position on the circle is set by $\beta_\theta^{(1)}$, which is of order $\bar\lambda/N_2^2$, and the two-loop terms shift it by about $50\bar\lambda$, a large coefficient. Along the family this shift, approximately $0.6/n$, vanishes, as do all higher-loop corrections relative to the leading term. The ordered fixed points of the family are therefore under parametric perturbative control.

The thermal state.—At the ordered fixed point, the leading-order thermal effective potential can be minimized exactly. Since $h_i > 0$, at fixed $\mathrm{tr}\,\Phi_i^\dagger\Phi_i$ the single-trace term is smallest when all eigenvalues of $\Phi_i^\dagger\Phi_i$ are equal. For $c_2 < 0 < c_1$ the global minimum is then

$$\Phi_1 = 0\,, \qquad \Phi_2 = v\,U\,, \qquad v^2 = -\frac{c_2N_2}{2a_2}\,T^2\,, \quad (11)$$

with $U \in U(N_2)$, provided that $c_1N_1a_2 > \zeta c_2N_2$, which holds with a wide margin. Relative to the symmetric point the free energy density is $\Delta f_* = -4\pi^2c_2^2N_2^2T^4/a_2$, and along the family we find $\Delta f_* \to -0.0029673\,n^5T^4$.

The condensate breaks $U(1)_{\Phi_2}$ to $\mathbb{Z}_{N_2}$. A rotation $e^{i\alpha}$ can be undone by a gauge transformation only if $e^{iN_2\alpha} = 1$, since gauge transformations leave $\det\Phi_2$ invariant. Accordingly $\langle\det\Phi_2\rangle_T = v^{N_2}\det U \neq 0$, and

the phase of $\det\Phi_2$ is a gauge-invariant Goldstone boson of the three-dimensional theory of static modes. The gauge group is Higgsed to $SU(N_2)_{\rm diag}$, as in color–flavor locking [38], but here both factors are gauged, and the only broken global symmetry is $U(1)_{\Phi_2}$. The thermal state is a relativistic superfluid [39], with stiffness $2v^2/(N_2T) \simeq 0.066\, n^2T$.

Beyond leading order.—The ordered minimum results from a competition between the tree-level potential and one-loop thermal masses, which might be undone by higher orders as explained in the introduction. This does not happen here. At the minimum, the masses in the small sector grow like $\sqrt{n}\,T$ (for instance, the Higgsed vectors have $M_W^2 \simeq 0.127\, nT^2$), but its contributions to c_2 are suppressed by r relative to those of the large sector. The $2n^6$ real scalars of the large sector stay light, with thermal mass $M_{1,0} \simeq 0.97\, T/\sqrt{n}$ at $\Phi_2 = 0$, so the leading correction to Δf_* comes from their static modes. Their free energy $-2N_1^2TM_1^3/(12\pi)$ depends on the condensate through $M_1^2 = M_{1,0}^2 + 32\pi^2\zeta v^2/N_1$, and including it gives

$$\Delta f_* = -0.0029673\, n^5T^4\Big[1 - \frac{1.817}{\sqrt{n}} + O\Big(\frac{\log n}{n}\Big)\Big]\,, \quad (12)$$

so the correction is suppressed by $n^{-1/2}$. All other corrections are suppressed by further powers of $1/n$, up to logarithms (End Matter).

Corrections could also change the shape of the condensate. A condensate of rank $N_2 - 1$, for which $\det\Phi_2 = 0$, differs in free energy from the full-rank one only by a relative amount of order $1/N_2$, which is smaller than the correction above. But the large sector couples to Φ_2 only through $\operatorname{tr}\Phi_2^\dagger\Phi_2$, so integrating it out, to all orders in its own interactions, cannot distinguish condensates with the same trace. The single-trace term $\tilde{h}_2 \operatorname{tr}(\Phi_2^\dagger\Phi_2)^2$ then gives every stationary point of rank $0 < k < N_2$ a direction of negative curvature, $-2\tilde{h}_2 v_k^2$, where v_k^2 is the nonzero eigenvalue of $\Phi_2^\dagger\Phi_2$ there (End Matter).

Entropic order.—It remains to understand why heating does not restore the symmetry. In any homogeneous phase of a CFT the free energy density f is proportional to T^4, leading to entropy and energy densities $s = -4f/T$ and $e = -3f$. The ordered state therefore has both more entropy and more energy than the symmetric state, $\Delta s_* = -4\Delta f_*/T$ and $\Delta e_* = -3\Delta f_*$: order at all temperatures in a CFT is necessarily entropic. Which modes supply this entropy can be read off directly from c_2. Each bosonic mode of mass M contributes $T^2M^2/24$ to f, so c_2 records how the condensate shifts the masses of all modes: modes that become heavier contribute less entropy, and modes that become lighter add it. In Eq. (5), the terms $\frac{1}{2} - \frac{\widehat{R}}{6}\cos\theta$ come from the Higgsed vectors and the fluctuations of Φ_2, which become heavier. The term $\frac{\widehat{R}}{6}\sin\theta/r$, which is negative on the ordered arc, comes from the $2N_1^2$ real components of Φ_1, whose mass squared decreases by $32\pi^2|\zeta|v^2/N_1$. The shift is suppressed by $1/N_1$, but it is shared by N_1^2 modes, so it governs the small-r behavior, and the bound $r < 0.19$ counts entropy. The disordered large sector serves as an entropy reservoir for the small sector, which orders. A single irreducible scalar multiplet cannot play both roles, since its thermal mass at one loop is nonnegative whenever its potential is bounded from below: gauge and Yukawa contributions are positive, and by irreducibility the scalar contribution is proportional to the average of the quartic potential over field directions. The mechanism is reminiscent of the Pomeranchuk effect in helium-3, where the solid has more entropy than the liquid [40].

Superconductivity at parametrically high temperature.—A superfluid becomes a superconductor when its broken symmetry is gauged. We can gauge $U(1)_{\Phi_2}$ by coupling the theory to a dynamical photon with a small gauge coupling e. The condensate Higgses $U(1)_{\Phi_2}$, and static magnetic fields are screened by the Meissner effect. In an ordinary Abelian plasma, there is electric Debye screening, but the static transverse photon has no perturbative screening mass. In the ordered phase, the condensate instead gives the photon a Meissner mass,

$$m_{\rm M}^2 = 2e^2N_2v^2\,, \qquad \frac{m_{\rm M}}{T} \simeq 0.257\, e\, n^2\,, \quad (13)$$

where we used $v^2/T^2 \simeq 0.0331\, n^3$ along the family. We keep $m_{\rm M}/T$ of order one and the gauge coupling small by defining $e = \widehat{e}/n^2$ at a reference scale μ_0, with $\widehat{e}$ of order one. Then $m_{\rm M}/T \simeq 0.257\,\widehat{e}$ stays finite, while the photon loop parameter $N_2^2e^2/16\pi^2$ vanishes as n^{-2}. The photon does not disturb the ordered fixed point, because its contributions to the thermal mass of Φ_2 and the matter β-functions are subleading in n.

Unlike the matter sector, the weakly gauged theory is not conformal, since the gauge coupling runs according to $\frac{de}{d\log\mu} = \frac{N_2^2e^3}{48\pi^2}$. For $e(\mu_0) = \widehat{e}/n^2$, the corresponding Landau pole at $\Lambda_{\rm L} = \mu_0\exp\Big(\frac{24\pi^2n^2}{\widehat{e}^2}\Big)$ marks the breakdown of the weakly coupled description. Far below $\Lambda_{\rm L}$, the running is parametrically slow. Between μ_0 and $\mu_0\exp(n)$, the gauge coupling changes only by $\frac{\Delta e}{e} = \frac{\widehat{e}^2}{48\pi^2 n} + O(n^{-2})$. The description remains controlled throughout this exponentially large range, and each gauged $\mathcal{T}_n$ remains in the superconducting phase. Superconductivity from entropic order was previously proposed in a large-N scalar model below its cutoff [14]; here the ordered sector is a CFT, and only the photon has a Landau pole.

Summary.—The circle of planar fixed points of CCR is lifted at order $1/N^2$, and the ranks and numbers of flavors can be chosen such that a fixed point remains on the ordered arc. Since all couplings along the family $\mathcal{T}_n$ are of order $1/n$, the existence of the fixed point and the superfluidity in its thermal state are under parametric perturbative control. To our knowledge, the theories $\mathcal{T}_n$, together with the theory of Komargodski and Popov [32],

are the first unitary, local, four-dimensional CFTs with finitely many fields whose thermal state breaks a continuous global symmetry spontaneously at every temperature. In addition, coupled to a weakly coupled photon, the same theories are superconductors over an exponentially large range of temperatures. Ordered fixed points also exist in much smaller theories, such as the two examples above, but their thermal corrections are under far less control. The fixed points are multicritical: besides the two mass terms, three combinations of the quartic couplings are weakly relevant, so reaching them from a generic ultraviolet theory requires tuning.

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[1] D. A. Kirzhnits and A. D. Linde, *Macroscopic consequences of the Weinberg model*, Phys. Lett. B **42**, 471 (1972).

[2] L. Dolan and R. Jackiw, *Symmetry behavior at finite temperature*, Phys. Rev. D **9**, 3320 (1974).

[3] S. Weinberg, *Gauge and global symmetries at high temperature*, Phys. Rev. D **9**, 3357 (1974).

[4] R. N. Mohapatra and G. Senjanović, *Broken symmetries at high temperature*, Phys. Rev. D **20**, 3390 (1979).

[5] P. Langacker and S.-Y. Pi, *Magnetic monopoles in grand unified theories*, Phys. Rev. Lett. **45**, 1 (1980).

[6] G. R. Dvali, A. Melfo, and G. Senjanović, *Is there a monopole problem?*, Phys. Rev. Lett. **75**, 4559 (1995), hep-ph/9507230.

[7] P. Meade and H. Ramani, *Unrestored electroweak symmetry*, Phys. Rev. Lett. **122**, 041802 (2019), arXiv:1807.07578.

[8] J. Orloff, *The UV price for symmetry nonrestoration*, Phys. Lett. B **403**, 309 (1997), hep-ph/9611398.

[9] G. Bimonte and G. Lozano, *On symmetry nonrestoration at high temperature*, Phys. Lett. B **366**, 248 (1996), hep-th/9507079.

[10] T. G. Roos, *Wilson renormalization group study of inverse symmetry breaking*, Phys. Rev. D **54**, 2944 (1996), hep-th/9511073.

[11] M. B. Pinto and R. O. Ramos, *A nonperturbative study of inverse symmetry breaking at high temperatures*, Phys. Rev. D **61**, 125016 (2000), hep-ph/9912273.

[12] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici, and M. Smolkin, *Thermal order in conformal theories*, Phys. Rev. D **102**, 065014 (2020), arXiv:2005.03676.

[13] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici, and M. Smolkin, *Symmetry breaking at all temperatures*, Phys. Rev. Lett. **125**, 131603 (2020).

[14] Y. Han, X. Huang, Z. Komargodski, A. Lucas, and F. K. Popov, *Entropic order*, Nature Commun. **17**, 87 (2026), arXiv:2503.22789.

[15] Z. Komargodski and F. K. Popov, *Temperature-resistant order in 2+1 dimensions*, Phys. Rev. Lett. **135**, 091602 (2025), arXiv:2412.09459.

[16] M. Hogervorst, S. Rychkov, and B. C. van Rees, *Unitarity violation at the Wilson–Fisher fixed point in $4-\epsilon$ dimensions*, Phys. Rev. D **93**, 125025 (2016), arXiv:1512.00013.

[17] N. Chai, A. Dymarsky, and M. Smolkin, *Model of persistent breaking of discrete symmetry*, Phys. Rev. Lett. **128**, 011601 (2022), arXiv:2106.09723.

[18] N. Chai, A. Dymarsky, M. Goykhman, R. Sinha, and M. Smolkin, *A model of persistent breaking of continuous symmetry*, SciPost Phys. **12**, 181 (2022), arXiv:2111.02474.

[19] E. Witten, *Chiral symmetry, the $1/N$ expansion, and the $SU(N)$ Thirring model*, Nucl. Phys. B **145**, 110 (1978).

[20] E. Rabinovici and M. Smolkin, *On the dynamical generation of the Maxwell term and scale invariance*, JHEP **07**, 040 (2011), arXiv:1102.5035.

[21] S. R. Coleman, *There are no Goldstone bosons in two dimensions*, Commun. Math. Phys. **31**, 259 (1973).

[22] B. Hawashin, J. Rong, and M. M. Scherer, *Ultraviolet-complete local field theory of persistent symmetry breaking in $2+1$ dimensions*, Phys. Rev. Lett. **134**, 041602 (2025), arXiv:2409.10606.

[23] N. Chai, E. Rabinovici, R. Sinha, and M. Smolkin, *The bi-conical vector model at $1/N$*, JHEP **05**, 192 (2021), arXiv:2011.06003.

[24] M. Smolkin and L. Yung, *Thermal order in the biconical model*, arXiv:2608.02720 (2026).

[25] M. Smolkin and L. Yung, *Persistent spontaneous time-reversal breaking*, arXiv:2609.16162 (2026).

[26] S. Chaudhuri, B. Hawashin, E. Rabinovici, and M. M. Scherer, *Exploring thermal order in conformal theories with multiple scalars coupled to an $O(N)$ vector field*, arXiv:2609.00160 (2026).

[27] S. Chaudhuri, C. Choi, and E. Rabinovici, *Thermal order in large N conformal gauge theories*, JHEP **04**, 203 (2021), arXiv:2011.13981.

[28] S. Chaudhuri and E. Rabinovici, *Symmetry breaking at high temperatures in large N gauge theories*, JHEP **08**, 148 (2021), arXiv:2106.11323.

[29] Asymptotically free theories with a scalar whose thermal mass is negative at arbitrarily high temperature have also been found [30, 31], but there the scalar is a single fundamental of its gauge group, and its condensate Higgses the gauge group without breaking a global symmetry.

[30] B. Bajc, A. Lugo, and F. Sannino, *Asymptotically free and safe fate of symmetry nonrestoration*, Phys. Rev. D **103**, 096014 (2021), arXiv:2012.08428.

[31] B. Bajc, G. Muco, F. Sannino, and S. Wagner, *Infinite heat order in 3+1 dimensions*, Phys. Rev. D **113**, 125029 (2026), arXiv:2604.01184.

[32] Closely related results were obtained independently by Z. Komargodski and F. K. Popov, including a theory in which an $SU(3)_L \times SU(3)_R$ chiral symmetry is broken at all temperatures. A joint discussion of their results and ours is given in Ref. [33].

[33] S. Harder, Z. Komargodski, F. K. Popov, A. Sharon, and Claude, *Resisting Misalignment*, Bhaumik Institute blog post (October 2026), https://bhaumik-institute.physics.ucla.edu/sites/default/files/news/resisting-misalignment/index.html.

[34] W. E. Caswell, *Asymptotic behavior of non-Abelian gauge theories to two-loop order*, Phys. Rev. Lett. **33**, 244

(1974).
[35] T. Banks and A. Zaks, *On the phase structure of vector-like gauge theories with massless fermions*, Nucl. Phys. B **196**, 189 (1982).
[36] A. Dymarsky, I. R. Klebanov, and R. Roiban, *Perturbative search for fixed lines in large N gauge theories*, JHEP **08**, 011 (2005), hep-th/0505099.
[37] E. Pomoni and L. Rastelli, *Large N field theory and AdS tachyons*, JHEP **04**, 020 (2009), arXiv:0805.2261.
[38] M. G. Alford, K. Rajagopal, and F. Wilczek, *Color flavor locking and chiral symmetry breaking in high density QCD*, Nucl. Phys. B **537**, 443 (1999), hep-ph/9804403.
[39] D. T. Son, *Low-energy quantum effective action for relativistic superfluids*, hep-ph/0204199 (2002).
[40] I. Pomeranchuk, *On the theory of liquid He^3*, Zh. Eksp. Teor. Fiz. **20**, 919 (1950).
[41] M. E. Machacek and M. T. Vaughn, *Two loop renormalization group equations in a general quantum field theory. 3. Scalar quartic couplings*, Nucl. Phys. B **249**, 70 (1985).
[42] M.-x. Luo, H.-w. Wang, and Y. Xiao, *Two loop renormalization group equations in general gauge field theories*, Phys. Rev. D **67**, 065019 (2003), hep-ph/0211440.
[43] L. Sartore and I. Schienbein, *PyR@TE 3*, Comput. Phys. Commun. **261**, 107819 (2021), arXiv:2007.12700.

END MATTER

Finite-N β-functions

With $\epsilon_i = 1/N_i^2$, $\epsilon_{12} = 1/(N_1 N_2)$, and $j \neq i$, the β-functions are as follows, exact in N_i. Through two loops the gauge couplings obey

$$\beta_{\lambda_i} = -\frac{21-4x_i}{3}\,\lambda_i^2 + \Big[\frac{26x_i-54}{3} - (4+2x_i)\,\epsilon_i\Big]\lambda_i^3\,, \quad (14)$$

whose zero is Eq. (2). The quartic couplings have $\beta = \beta^{1\mathrm{L}} + \beta^{2\mathrm{L}}$, with

$$\begin{aligned}
\beta^{1\mathrm{L}}_{h_i} &= 8h_i^2 - 12(1-\epsilon_i)\lambda_i h_i + \big(\tfrac{3}{2} - 12\epsilon_i\big)\lambda_i^2 + 24\epsilon_i f_i h_i\,,\\
\beta^{1\mathrm{L}}_{f_i} &= 4(1+4\epsilon_i)f_i^2 + 16 f_i h_i - 12(1-\epsilon_i)\lambda_i f_i + 12h_i^2\\
&\quad + \big(\tfrac{9}{2} + 6\epsilon_i\big)\lambda_i^2 + 4\zeta^2\,,\\
\beta^{1\mathrm{L}}_{\zeta} &= \zeta\big[4(1+\epsilon_1)f_1 + 4(1+\epsilon_2)f_2 + 8h_1 + 8h_2\\
&\quad - 6(1-\epsilon_1)\lambda_1 - 6(1-\epsilon_2)\lambda_2 + 8\epsilon_{12}\zeta\big]\,,
\end{aligned} \quad (15)$$

and

$$\begin{aligned}
\beta^{2\mathrm{L}}_{h_i} &= \big(4 - \tfrac{8}{3}x_i + \tfrac{2}{3}(15+32x_i)\epsilon_i - 120\epsilon_i^2\big)\lambda_i^3\\
&\quad + \big(\tfrac{20}{3}x_i - 26 - \tfrac{4}{3}(9+5x_i)\epsilon_i + 18\epsilon_i^2\big)\lambda_i^2 h_i\\
&\quad + 12\epsilon_i(1-8\epsilon_i)\lambda_i^2 f_i + 8(5-11\epsilon_i)\lambda_i h_i^2\\
&\quad + 96\epsilon_i(1-\epsilon_i)\lambda_i h_i f_i - 24(1+5\epsilon_i)h_i^3\\
&\quad - 352\epsilon_i h_i^2 f_i - 8\epsilon_i(5+41\epsilon_i)h_i f_i^2 - 40\epsilon_i h_i\zeta^2\,,\\
\beta^{2\mathrm{L}}_{f_i} &= \big(12 - 8x_i + \tfrac{2}{3}(51-16x_i)\epsilon_i + 60\epsilon_i^2\big)\lambda_i^3\\
&\quad + \big(\tfrac{20}{3}x_i - 14 + \tfrac{20}{3}(6-x_i)\epsilon_i + 58\epsilon_i^2\big)\lambda_i^2 f_i\\
&\quad + 4(9-4\epsilon_i)\lambda_i^2 h_i + 48(2-\epsilon_i)\lambda_i h_i^2\\
&\quad + 128(1-\epsilon_i)\lambda_i h_i f_i + 16(2+3\epsilon_i - 5\epsilon_i^2)\lambda_i f_i^2\\
&\quad + 32(1-\epsilon_j)\lambda_j\zeta^2 - 96h_i^3 - 8(5+41\epsilon_i)h_i^2 f_i\\
&\quad - 352\epsilon_i h_i f_i^2 - 24\epsilon_i(3+7\epsilon_i)f_i^3\\
&\quad - 40\epsilon_i f_i\zeta^2 - 32\epsilon_{12}\zeta^3\,,\\
\beta^{2\mathrm{L}}_{\zeta} &= \zeta\sum_i\Big[\big(\tfrac{10}{3}x_i - 7 + \tfrac{2}{3}(3-5x_i)\epsilon_i + 5\epsilon_i^2\big)\lambda_i^2\\
&\quad + 64(1-\epsilon_i)\lambda_i h_i + 32(1-\epsilon_i^2)\lambda_i f_i\\
&\quad - 20(1+\epsilon_i)h_i^2 - 80\epsilon_i h_i f_i - 20\epsilon_i(1+\epsilon_i)f_i^2\Big]\\
&\quad + \epsilon_{12}\zeta^2\sum_i\big[8(1-\epsilon_i)\lambda_i - 96h_i - 48(1+\epsilon_i)f_i\big]\\
&\quad - 4(\epsilon_1+\epsilon_2+8\epsilon_1\epsilon_2)\zeta^3\,.
\end{aligned} \quad (16)$$

Both contain only the corrections ϵ_i and ϵ_{12}, and no positive powers of N_1/N_2. The two-loop terms follow from the general two-loop results [41, 42], and we checked all of these β-functions against PyR@TE 3 [43] for several values of the ranks. Since $\beta_\zeta \propto \zeta$, the decoupled points $\zeta = 0$ are exact fixed points.

The β-function on the circle

Write the quartic couplings as $q = (h_1, h_2, f_1, f_2, \zeta)$ and split $\beta = \beta^{(0)} + \beta^{(1)}$, where $\beta^{(0)}$ is the planar part at $\lambda_1 = \lambda_2 = \bar\lambda$, and $\beta^{(1)}$ contains the terms with ϵ_i and ϵ_{12} and the terms due to δ. Near a point $q_0(\theta)$ of the circle, a shift $q = q_0 + \delta q$ changes $\beta^{(0)}$ by $J\,\delta q$, where J is its Jacobian. Exactly one combination of the β-functions is unchanged at this order, $\ell \cdot J = 0$, with

$$\ell(\theta) = \big(-p\sin\theta,\, p\sin\theta,\, -\sin\theta,\, \sin\theta,\, 2\cos\theta\big)\,, \quad (17)$$

and $p = (2+3\sqrt{6})/4$. All other combinations, the transverse β-functions, can be set to zero by a choice of δq, and the fixed-point condition reduces to $\ell \cdot \beta^{(1)}(q_0(\theta)) = 0$. Since $\ell \cdot \partial_\theta q_0 = 2\widehat{R}\bar\lambda$, the θ-component of $\beta^{(1)}$ is $\beta^{(1)}_\theta = \ell\cdot\beta^{(1)}(q_0(\theta))/(2\widehat{R}\bar\lambda)$, the leading term of $\beta_\theta = d\theta/d\log\mu$. Every simple zero of $\beta^{(1)}_\theta$ gives a fixed point near $q_0(\theta)$,

TABLE I. Fixed points of the one- and two-loop β-functions, exact in N_i. Here $\theta^{1\mathrm{L}}$ and $\theta^{2\mathrm{L}}$ are the angles $\theta = \arg[(f_1-f_2)/2+i\zeta]$ at one and two loops, $c_2^{2\mathrm{L}}$ is evaluated with the two-loop couplings, and D/D_{LO} is the depth of the ordered minimum including the correction from the static modes of the large sector, relative to its leading-order value.

theory	$10^3\lambda_1^*$	$10^3\lambda_2^*$	Δ	$\theta^{1\mathrm{L}}$	$\theta^{2\mathrm{L}}$	$c_2^{2\mathrm{L}}/\lambda_2^*$	D/D_{LO}
$(150, 783, 5, 26)$	1.468	2.517	-33.47	-1.844	-1.700	-0.793	0.221
$(300, 1562, 10, 52)$	2.130	2.476	-38.25	-1.957	-1.828	-1.79	0.261
$\mathcal{T}_5$	1.271	2.517	-41.85	-2.230	-2.143	-0.588	0.200
$\mathcal{T}_9$	1.156	1.368	-34.60	-1.796	-1.728	-5.76	0.477
$\mathcal{T}_{13}$	0.871	0.941	-33.20	-1.725	-1.678	-13.9	0.555
$\mathcal{T}_{21}$	0.564	0.580	-32.43	-1.687	-1.659	-39.2	0.641
$\mathcal{T}_{41}$	0.294	0.296	-32.06	-1.670	-1.655	-155	0.735
$\mathcal{T}_{101}$	0.120	0.120	-31.93	-1.664	-1.658	-951	0.827

by the implicit function theorem. Inserting (15) we find

$$\beta_\theta^{(1)} = \frac{\sin\theta}{2\widehat{R}}\Big\{\kappa\,\delta + \bar\lambda\Big[(\epsilon_2-\epsilon_1)\big(\kappa\cos^2\theta + a\big) + (\epsilon_1+\epsilon_2)\,b\cos\theta + \epsilon_{12}\,\kappa\sin 2\theta\Big]\Big\}, \tag{18}$$

with $\kappa = (18\sqrt{6}-39)/2 = 8\widehat{R}^2$, $a = \frac{93}{2} - \frac{27\sqrt{6}}{4} \approx 29.966$, and $b = \sqrt{6\sqrt{6}-13}\,\big(\frac{9\sqrt{3}}{2} - \frac{135\sqrt{2}}{8}\big) \approx -20.935$. This is Eq. (6) with

$$g_r(\theta) = (1-r^2)\big(\kappa\cos^2\theta + a\big) + (1+r^2)\,b\cos\theta + r\kappa\sin 2\theta\,. \tag{19}$$

Every term is of order $1/N^2$: the non-planar terms through ϵ_i and ϵ_{12}, and the mismatch through Δ. At the interacting zeros the slope $\partial_\theta\beta_\theta^{(1)} = \bar\lambda\sin^2\theta_*|2\kappa\cos\theta_* + b|/(2\widehat{R}N_2^2) > 0$ gives the dimension $4 + \partial_\theta\beta_\theta^{(1)}$ of the operator that moves the fixed point along the old circle.

Fixed points

Table I lists the fixed points of the one- and two-loop β-functions, exact in N_i, for the two examples and for members of the family (9). In all cases the potential is bounded below, c_1/λ_1^* lies between 0.48 and 0.56, $c_2 < 0$, and the leading-order minimum (11) is a condensate of full rank.

Thermal corrections

The leading correction to the thermal free energy comes from the static modes of the $2N_1^2$ real scalars of the large sector, whose mass in the background $\Phi_2 = v\mathbf{1}$ is $M_1^2 = M_{1,0}^2 + 32\pi^2\zeta v^2/N_1$. With $w = v^2/(N_1T^2)$ and the limiting ratios $\widehat{a} = a_2/\bar\lambda$, $\widehat{\zeta} = \zeta/\bar\lambda$, and $\widehat{c} = c_1/\bar\lambda$, the free energy density along the full-rank direction is

$$\frac{\Delta f(w)}{16\pi^2\bar\lambda N_1^2T^4} = \widehat{a}\,w^2 + \frac{\widehat{\zeta}}{6}\,w + \frac{2}{3}\sqrt{\bar\lambda}\Big[\widehat{c}^{\,3/2} - \big(\widehat{c} + 2\widehat{\zeta}w\big)^{3/2}\Big] + \cdots. \tag{20}$$

At leading order the minimum is at $w_0 = -\widehat{\zeta}/(12\widehat{a}) \approx 0.033116$, with depth $D_{\mathrm{LO}} = \widehat{a}\,w_0^2 \approx 1.5502\times10^{-3}$. The last term reduces the depth to $D/D_{\mathrm{LO}} = 1 - 16.503\sqrt{\bar\lambda} + O(\bar\lambda)$, which with $\bar\lambda \simeq 2/(165n)$ gives Eq. (12). The last column of Table I uses the exact two-loop couplings and includes the contributions of the small sector to c_2. The remaining corrections can be estimated by power counting. Loops of the small sector carry no factor of N_1/N_2 and are suppressed by $r = 1/n^2$; higher loops of the large sector are suppressed by $\bar\lambda$, up to logarithms of $M_W/T \sim \sqrt{n}$; and the soft modes of the unbroken gauge groups enter only through further loops. All are therefore of relative order $\log n/n$ or smaller.

For the shape of the condensate, let $G(S)$ collect all terms of the effective potential that depend on Φ_2 only through $S = \mathrm{tr}\,\Phi_2^\dagger\Phi_2$: the double-trace term, the thermal mass, and the contribution of the large sector to all orders in its interactions, so that $V_{\mathrm{eff}} = \tilde{h}_2\,\mathrm{tr}(\Phi_2^\dagger\Phi_2)^2 + G(S) +$ (small-sector loops). At a stationary point of rank k, with k equal nonzero eigenvalues v_k^2, stationarity gives $G'(S) = -2\tilde{h}_2 v_k^2$, which is the curvature in the direction of a vanishing eigenvalue, so every such point is a saddle, independently of G. The small-sector loops that could distinguish shapes are suppressed by $\bar\lambda$, up to logarithms, and carry no power of N_1/N_2.